Averaging in a nutshell

Ph. Chartier[*] M. Lemou[†] F. Méhats[‡] and L. Trémant[§]

January 13, 2022

Abstract

This paper presents averaging methods and their main properties in a new and very concise way.

Keywords: highly-oscillatory evolution equation, standard and stroboscopic averaging, geometric integration.

MSC numbers: 34K33, 37L05, 35Q55.

1 Introduction

This paper compiles results pertaining to *high-order averaging*, that is to say the problem of separating the slow and fast dynamics in a highly-oscillatory setting. The type of problem we consider may arise in many realistic physical models, such as molecular dynamics [GSS98] or charged-particle dynamics under a strong magnetic field [CCLMZ20; FSS09; FS00]. It may also arise in functional spaces; two examples are the nonlinear Klein-Gordon equation in the nonrelativistic limit regime [BCZ14; BZ19; CLMV20] and the oscillatory nonlinear Schrödinger equation [CCLM15; CCMM15].

Mathematically speaking, we consider problems with forced oscillations of the form

$$\partial_t u^\varepsilon(t) = f_{t/\varepsilon}\big(u^\varepsilon(t)\big), \qquad u^\varepsilon(0) = u_0 \in X, \qquad t \in [0,1] \tag{1.1}$$

where X is a Banach space of norm $|\cdot|$, the non-autonomous vector field $(\theta, u) \in \mathbb{T} \times X \mapsto f_\theta(u)$ is 1-periodic w.r.t. θ on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. As mentioned, the space X may be simply $\mathbb{R}^d$, in which case the problem is a simple ordinary differential equation in finite dimension, or it may be a functional space, such as the space of square-integrable function $L^2(\mathbb{R})$. Note that this type of equation can result from the *filtering* of an autonomous equation

$$\dot v^\varepsilon = \frac{1}{\varepsilon} G(v^\varepsilon) + K(v^\varepsilon), \quad v^\varepsilon(0) = v_0 \in X \tag{1.2}$$

if G generates a 1-periodic flow $(\theta, u) \mapsto \chi_\theta(u)$. It links to 1.1 using the filtered variable $u^\varepsilon(t) = \chi_{-t/\varepsilon}\big(v^\varepsilon(t)\big)$ which follows an equation of the form (1.1) with $f_\theta(u) = (\partial_u \chi_{-\theta} \cdot K) \circ \chi_\theta(u)$.

[*]INRIA-Rennes, IRMAR and ENS Bruz, Campus de Beaulieu, 35042 Rennes Cedex, France. Email: Philippe.Chartier@inria.fr

[†]CNRS, IRMAR, Université de Rennes 1 and INRIA-Rennes, Campus de Beaulieu, 35042 Rennes Cedex, France. Email: Mohammed.Lemou@univ-rennes1.fr

[‡]IRMAR, Université de Rennes 1 and INRIA-Rennes, Campus de Beaulieu, 35042 Rennes Cedex, France. Email: Florian.Mehats@univ-rennes1.fr

[§]Univ Rennes 1, Inria Rennes and IRMAR, Campus de Beaulieu, 35042 Rennes Cedex, France. Email: Leopold.Tremant@inria.fr

The approach of averaging can be summarized as the decomposition of the solution $u^\varepsilon(t)$ into a *near-identity, rapidly oscillating* change of variable $\Phi^\varepsilon_{t/\varepsilon}$ and the dynamics of an *average* autonomous vector field F^ε. This can be written

$$u^\varepsilon(t) = \Phi^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_t \circ \left(\Phi^\varepsilon_0\right)^{-1}(u_0), \tag{1.3}$$

where $(\theta, u) \mapsto \Phi^\varepsilon_\theta(u)$ is 1-periodic w.r.t. θ and $(t, u) \mapsto \Psi^\varepsilon_t(u)$ is the t-flow associated to F^ε, i.e. for $(t, u) \in [0, 1] \times X$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi^\varepsilon_t(u) = F^\varepsilon\left(\Psi^\varepsilon_t(u)\right), \qquad \Psi^\varepsilon_0 = \mathrm{id}\,. \tag{1.4}$$

We refer to Lochak-Meunier [LM88] and Sanders-Verhulst-Murdock [SVM07] for textbooks on these issues. Since the goal is to separate the fast periodic part in θ and the slow drift in t, averaging can be seen as analogous to the two-scale expansion $u^\varepsilon(t) = U^\varepsilon(t, \theta)|_{\theta=t/\varepsilon}$ often found in the context of high-frequency PDEs. It is also similar to WKB expansions [Wen26; Kra26; Bri26], since in some sense Φ^ε captures the rapid phase dynamics and Ψ^ε the slow amplitude changes.

A particularly well-studied case is that of the autonomous problems with linearly-generated oscillations (i.e. linear G), for which the problem of averaging can often be reduced to finding some θ-independent change of variable $\left(\Phi^\varepsilon_0\right)^{-1}$, or some equivalent. It is then possible to consider the problem on this new variable $\left(\Phi^\varepsilon_0\right)^{-1}(u(t))$. As such, averaging in this context is linked to normal forms, in particular Birkhoff's forms have been considered in this context by Bambusi [Bam03; BB05; Bam06; Bam08], Bourgain [Bou96], Colliander [CKSTT10; CKO12] and Grébert [Bam06; GV11; GT12], to mention only a few. Note that many of these works on normal forms consider the setting of multiple non-resonant frequencies, which is akin to considering f as a function of multiple phases $\theta_1, \theta_2, \ldots$ in (1.1). This setting has also been studied with averaging using diophantine approximations in [CMTZ17] and with B-series in [CMS12b].

In this work, we shall not discuss specific methods to compute the periodic change of variable or the averaged vector fields, the traditional approach dating back to [Per69] consists in assuming the maps are power series in ε and injecting the ansatz $\Phi^\varepsilon_\theta = \mathrm{id} + \sum_{n\geq 1} \Phi^{[n]}_\theta$ in (1.3) and identifying like terms in ε. This formal series approach has been revisited using B-series or the Magnus expansion in [CMS10; CMS12a; CCM19]. Another approach is that of "successive substitution" dating back to [Nei84] (albeit in a slightly different context), and more recently in [CCMM15; CLMV20]. This circumvents the ansatz and yields an *exponential* error, i.e. an error bounded by $Ce^{-\nu/\varepsilon}$ for some $C > 0$ and $\nu > 0$. Both approaches coincide formally. Our goal in this paper is to present known results under a new light, and to offer original proofs without having to invoke any ansatz, formal series or construction process.

In Section 2, we present some general properties of averaging, detailing the differences between standard and stroboscopic averaging. In Section 3, we enounce two assumptions which describe what we mean by exact and approximate averaging. In Section 4, we present some remarkable properties of averaging in the autonomous case, namely that the change of variable is the flow of an autonomous problem, and it commutes with Ψ^ε. In Section 5, we restrain ourselves to stroboscopic averaging, and show that it preserves geometric properties. For each context we consider different assumptions, summarized by the following table

	autonomous	geometry
exact	✓	
linear exact	✓	✓
approximate	✓	✓

In the context of approximate averaging, the properties of averaging are shown up to an error of the same order as the error of approximation.

2 A brief presentation of averaging

Differentiating (1.3) w.r.t. t generates

$$f_{t/\varepsilon} \circ \Phi^\varepsilon_{t/\varepsilon}\big(v(t)\big) = \frac{1}{\varepsilon}\partial_\theta \Phi^\varepsilon_{t/\varepsilon}\big(v(t)\big) + \partial_u \Phi^\varepsilon_{t/\varepsilon}\big(v(t)\big) \cdot F^\varepsilon\big(v(t)\big)$$

with $v(t) = \Psi^\varepsilon_t \circ \big(\Phi^\varepsilon_0\big)^{-1}(u_0)$ the average dynamics. By separating the rapid oscillations in t/ε and the slow drift in t, one obtains the *homological equation*, which is for $(\theta, u) \in \mathbb{T} \times X$,

$$\partial_\theta \Phi^\varepsilon_\theta(u) = \varepsilon \left(f_\theta \circ \Phi^\varepsilon_\theta(u) - \partial_u \Phi^\varepsilon_\theta(u)\, F^\varepsilon(u)\right). \tag{2.1}$$

Now taking the average, it appears that the change of variable Φ^ε alone stores the information of the averaged vector field. Indeed, for u in X, $F^\varepsilon(u)$ is given by

$$F^\varepsilon(u) = \Big(\partial_u \langle \Phi^\varepsilon \rangle(u)\Big)^{-1} \langle f \circ \Phi \rangle(u), \tag{2.2}$$

where $\langle \cdot \rangle$ denotes the average, defined for a periodic map $(\theta, u) \in \mathbb{T} \times X \mapsto \varphi_\theta(u)$ by

$$\langle \varphi \rangle(u) = \int_0^1 \varphi_\theta(u) \mathrm{d}\theta. \tag{2.3}$$

Up to a change of variable, Φ^ε is assumed to be near identity, i.e.

$$\Phi^\varepsilon = \mathrm{id} + \mathcal{O}(\varepsilon). \tag{2.4}$$

It is known that equation (2.1) generally has no rigorous solution, only solutions as a formal series in ε. An example where this divergence is observed can be found in [CMS10]. However the series converges in the case where f_θ is a linear and bounded operator, for ε small enough.

Perhaps the most straighforward approach to solve the homological equation is a fixed point method separating the right-hand side of the equation (of size ε) and the left (of size 1). It immediately appears that a closure condition on Φ^ε is needed to properly invert ∂_θ. Two choices are often considered.

Standard averaging: $\langle \Phi^\varepsilon \rangle = \mathrm{id}$,
: which circumvents the computation of an inverse, as then $F^\varepsilon = \langle f \circ \Phi^\varepsilon \rangle$, thereby making computations less costly. In this case, the method shares similarities with the so-called Chapmann-Enskog method in the context of kinetic theory (see [CCLM20]). As highlighted in [CLMZ20], in numerical contexts the $\partial_u \Phi^\varepsilon \cdot F^\varepsilon$-term can be replaced by a finite-differences approximation up to some order in ε, which removes the need to compute an exact derivative and makes automatic computations much simpler.

Stroboscopic averaging: $\Phi^\varepsilon_0 = \mathrm{id}$,
: for which the solution $u(t)$ coincides with the average $\Psi^\varepsilon_t(u_0)$ at "stroboscopic" times $t \in \varepsilon\mathbb{N}$. This produces more complex computations but allows for the preservation of geometric properties, such as energy conservation or symplectic structure.

We shall mainly focus on the properties of stroboscopic averaging in the upcoming sections, but it is important to keep in mind that these choices are conjugate. Indeed, the latter can be obtained from the former by setting

$$\Phi^{strob} = \Phi^{std} \circ \big(\Phi^{std}_0\big)^{-1} \quad \text{and} \quad \Psi^{strob} = \Phi^{std}_0 \circ \Psi^{std} \circ \big(\Phi^{std}_0\big)^{-1},$$

i.e. $F^{strob} = \big(\partial_u \Phi^{std}_0 \cdot F^{std}\big) \circ \big(\Phi^{std}_0\big)^{-1}$. Conversely, standard averaging can be obtained from stroboscopic averaging with the relations

$$\Phi^{std} = \Phi^{strob} \circ \big\langle \Phi^{strob} \big\rangle^{-1} \quad \text{and} \quad \Psi^{std} = \big\langle \Phi^{strob} \big\rangle \circ \Psi^{strob} \circ \big\langle \Phi^{strob} \big\rangle^{-1}. \tag{2.5}$$

Thus some properties of standard averaging will also be discussed.

3 Mathematical setting

We consider two different settings : one of *exact* averaging, and an other of *approximate* averaging. The first is on the entire space X and corresponds to the behaviour of linear problems. The second is on a possibly-bounded subspace $\mathcal{K}$ and corresponds to the behaviour of analytic problems. The results are stronger in the first setting, but the second is more general, therefore we treat them separately in the sequel.

Assumption 3.1 (Exact averaging)**.** *There exists an upper bound $\varepsilon_0 > 0$ such that for all $u_0 \in X$ and $\varepsilon \in (0, \varepsilon_0]$, Problem (1.1) is well-posed for $t \in [0, 1]$. Furthermore for all $\varepsilon \in (0, \varepsilon_0]$, the averaging maps $(\theta, u) \in \mathbb{T} \times X \mapsto \Phi^\varepsilon_\theta(u)$ and $u \in X \mapsto F^\varepsilon(u)$ are well-defined and smooth w.r.t. u. In addition, the t-flow $(t, u) \mapsto \Psi^\varepsilon_t(u)$ is well-defined for $t \in [0, 1]$ and the change of variable $u \mapsto \Phi^\varepsilon_\theta(u)$ is invertible for all $\theta \in \mathbb{T}$. The linear map $\partial_u \Phi^\varepsilon_\theta(u)$ is invertible for all $(\theta, u) \in \mathbb{T} \times X$.*

This setting corresponds to the behaviour of problems where $(\theta, u) \mapsto f_\theta(u)$ is linear w.r.t. u. It follows from this assumption that the t-flow of F^ε, namely $(t, u) \mapsto \Psi^\varepsilon_t(u)$ is defined for all $t \in \mathbb{R}$. Note that this assumption does not involve the closure condition on Φ^ε.

Before discussing the other setting, if X is a *real* Banach space, let us introduce $X_\mathbb{C}$ the complexification of X, defined as

$$X_\mathbb{C} := \{z = u + iv,\ (u, v) \in X^2\}.$$

We denote $u = \Re(z) \in X$ and $v = \Im(z) \in X$ the real and imaginary parts of z respectively. The complexified space $X_\mathbb{C}$ is a Banach space when endowed with the norm

$$|z|_{X_\mathbb{C}} := \sup_{|\xi|=1} |\Re(\xi z)|.$$

Note that for $u \in X$, this norm $|u|_{X_\mathbb{C}}$ coincides with the norm of the real space $|u|$, thus in the sequel, we write $|\cdot|$ to denote the norm on the complexified space $X_\mathbb{C}$ indiscriminately. Denote now $\mathcal{K}$ a subset of X on which we shall conduct our study. Given a radius $\rho > 0$, we denote

$$\mathcal{K}_\rho := \big\{u + \tilde{u}, \quad (u, \tilde{u}) \in \mathcal{K} \times X_\mathbb{C}, |\tilde{u}| \leq \rho\big\},$$

i.e. $\mathcal{K}_\rho$ is the extension of $\mathcal{K}$ in $X_\mathbb{C}$ by a radius ρ. We also define the norm, given a bounded map φ from $\mathcal{K}_\rho$ to some Banach space $(E, |\cdot|)$,

$$\|\varphi\|_\rho = \sup_{u \in \mathcal{K}_\rho} |\varphi(u)|.$$

In particular for maps $\varphi : X \to X$, we have $E = X$, and for their derivatives $E = \mathcal{L}(X, X)$ endowed with the operator norm $|||\cdot|||$.

Assumption 3.2 (Approximate averaging)**.** *There exists an upper parameter ε_0 such that Problem (1.1) is well-posed for $t \in [0, 1]$ with an initial condition $u(0) \in \mathcal{U}_0 \subset \mathcal{K}$, and X is a simply connected set. For some radius $R > 0$, the vector field $(\theta, u) \mapsto f_\theta(u)$ and its derivative are bounded (uniformly w.r.t. θ) on $\mathcal{K}_{3R}$. Given a rank $n \in \mathbb{N}$, for $\varepsilon \leq \varepsilon_n := \varepsilon_0/(n + 1)$, there are approximations $(\theta, u) \in \mathbb{T} \times \mathcal{K}_{3R} \mapsto \Phi^{[n]}_\theta(u)$ and $u \in \mathcal{K}_{3R} \mapsto F^{[n]}(u)$ which are well-defined and respect the bounds*

$$\sup_{\theta \in \mathbb{T}} \|\Phi^{[n]}_\theta - \mathrm{id}\,\|_{3R} \leq \frac{\varepsilon}{\varepsilon_n} \frac{R}{2} \qquad \textit{and} \qquad \|F^{[n]}\|_{3R} \leq M$$

for some $M > 0$ independent of ε. Both maps are analytic w.r.t. u, with a radius of convergence everywhere greater than R. Additionally, the error of approximation is of size ε^{n+1}, i.e. writing $\Psi_t^{[n]}$ the t-flow of $F^{[n]}$, for all $u_0 \in \mathcal{U}_0$ and $t \in [0,1]$,

$$\left| u(t) - \Phi^{[n]}_{t/\varepsilon} \circ \Psi^{[n]}_t \circ \left(\Phi^{[n]}_0\right)^{-1}(u_0) \right| \leq C \left(\frac{\varepsilon}{\varepsilon_n}\right)^{n+1} \tag{3.1}$$

with C independent of n, ε and t.

This assumption matches the behaviour generally observed with averaging when assuming $(\theta, u) \mapsto f_\theta(u)$ analytic w.r.t. u, found for instance in [CCMM15], and is sufficient to have $\Phi^{[n]}_\theta$, $\langle \Phi^{[n]} \rangle$ and their derivatives invertible for all $\theta \in \mathbb{T}$. In addition, as noted in [CMS15], if such approximations exist for all $n \in \mathbb{N}$, this is enough to ensure the historical optimal "exponential" error bound of [Nei84], which can be stated as such: there exist two positive constants C and ν such that for all $\varepsilon > 0$ one can choose an integer n *depending on* ε, such that for all t,

$$\left| u(t) - \Phi^{[n]}_{t/\varepsilon} \circ \Psi^{[n]}_t \circ \left(\Phi^{[n]}_0\right)^{-1}(u_0) \right| \leq C e^{-\nu/\varepsilon}.$$

This reflects the fact that the maps Φ^ε and F^ε can only be obtained as diverging power series in ε, therefore the error is *formal*, or up to a flat function. Indeed, in order to increase the order of the approximation, ε must get smaller and smaller, such that an error $\mathcal{O}(\varepsilon^\infty)$ is impossible with $\varepsilon \neq 0$. The error of approximation is caracterised by the defect $\delta^{[n]}$ defined by

$$\delta^{[n]}_\theta = \frac{1}{\varepsilon} \partial_\theta \Phi^{[n]}_\theta - f_\theta \circ \Phi^{[n]}_\theta + \partial_u \Phi^{[n]}_\theta \cdot F^{[n]},$$

which corresponds to the error in the homological equation (2.1). The previous assumptions corresponds to the situation

$$\sup_{\theta \in \mathbb{T}} \|\delta^{[n]}\|_{3R} = \mathcal{O}(\varepsilon^n) \qquad \text{and} \qquad \langle \delta^{[n]} \rangle = \mathcal{O}(\varepsilon^{n+1}). \tag{3.2}$$

4 Commutation of flows in the autonomous case

In this section we restrict ourselves to the case of an autonomous equation of the form

$$\dot{v}^\varepsilon = \frac{1}{\varepsilon} G(v^\varepsilon) + K(v^\varepsilon), \quad v^\varepsilon(0) = v_0 \in X \tag{4.1}$$

where G and K are smooth function from a Banach space X into itself and where G generates a 1-periodic flow $(\theta, u) \mapsto \chi_\theta(u)$. The approach is the same as for the non-autonomous problem, which is to say we search a solution under the form

$$v^\varepsilon(t) = \Omega^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_t \circ (\Omega^\varepsilon_0)^{-1}(v_0) \tag{4.2}$$

where $\theta \mapsto \Omega^\varepsilon_\theta$ is assumed to be 1-periodic and Ψ^ε_t is the t-flow associated to the averaged vector flow K^ε. The reasons why the notation of the change of variable changed but not that of the average flow will be made clear as this section progresses. The homological equation is now

$$\partial_\theta \Omega^\varepsilon_\theta(u) - G \circ \Omega^\varepsilon_\theta(u) = \varepsilon \Big(K \circ \Omega^\varepsilon_\theta(u) - \partial_u \Omega^\varepsilon_\theta(u)\, K^\varepsilon(u) \Big). \tag{4.3}$$

It appears that the closure condition of standard averaging must be reconsidered. Indeed, in the limit $\varepsilon \to 0$, the change of variable $\Omega^\varepsilon_\theta$ approaches $\chi_{\theta+\theta_0}$ for some initial phase θ_0. Consider for instance the

case $G(u) = 2\pi \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} = 2\pi J u$, then clearly choosing the standard closure condition $\langle \Omega^\varepsilon \rangle = \mathrm{id}$ cannot hold, as $\langle \chi \rangle = 0$. Rather than discarding standard averaging altogether, we may *filter* the equation, which is to say transform it into a problem of the form (1.1) (i.e. with forced oscillations) by left-multiplying it by $\partial_u \chi_{-\theta+\theta_1}\big(\Omega^\varepsilon_\theta\big)$ for some arbitrary phase θ_1. Define the filtered change of variable $\Phi^\varepsilon_{\theta,\theta_1} = \chi_{-\theta+\theta_1} \circ \Omega^\varepsilon_\theta$, it satisfies[1]

$$\partial_\theta \Phi^\varepsilon_{\theta,\theta_1}(u) = \varepsilon\Big(f_{\theta,\theta_1} \circ \Phi^\varepsilon_{\theta,\theta_1}(u) - \partial_u \Phi^\varepsilon_{\theta,\theta_1}(u)\, K^\varepsilon(u) \Big) \tag{4.4}$$

with $f_{\theta,\theta_1}(u) = (\partial_u \chi_{-\theta+\theta_1} \cdot K) \circ \chi_{\theta-\theta_1}(u)$. Note that we exploited the identity $\partial_\theta \chi_\theta = G \circ \chi_\theta = \partial_u \chi_\theta\, G$. Take now the average on θ of (4.4),

$$0 = \varepsilon\Big(\big\langle f_{\bullet,\theta_1} \circ \Phi_{\bullet,\theta_1} \big\rangle(u) - \partial_u \big\langle \Phi_{\bullet,\theta_1} \big\rangle(u)\, K^\varepsilon(u) \Big). \tag{4.5}$$

The standard choice of closure condition is therefore $\langle \Phi_{\bullet,\theta_1} \rangle = \mathrm{id}$, i.e. $\Omega^\varepsilon_\theta$ close to $\chi_{\theta-\theta_1}$. Remember however that the phase shift θ_1 is arbitrary, therefore there are infinitely many standard closure conditions, the canonical one being $\langle \chi_{-\theta} \circ \Omega^\varepsilon_\theta \rangle = \mathrm{id}$.

4.1 The case of exact averaging

Assuming that $\Omega^\varepsilon_\theta$ is close to $\chi_{\theta+\theta_0}$, all filtered changes of variable satisfy

$$\Phi^\varepsilon_{\theta,\theta_1} = \chi_{-\theta+\theta_1} \circ \big(\chi_{\theta+\theta_0} + \mathcal{O}(\varepsilon) \big) = \chi_{\theta_1+\theta_0} + \mathcal{O}(\varepsilon).$$

Consider now that the averaging maps of the filtered homological equation (4.4), namely $\Phi^\varepsilon_{\theta,\theta_1}$ and K^ε, satisfy Assumption 3.1 up to replacing id by $\chi_{\theta_1+\theta_0}$, and assume furthermore that the operator $\partial_u \chi_\theta(u)$ is uniformly bounded for all $(\theta, u) \in \mathbb{T} \times X$. Then $\partial_u \langle \Phi^\varepsilon_{\theta,\theta_1} \rangle$ is invertible for ε small enough, and we may write from (4.5),

$$K^\varepsilon(u) = \Big(\partial_u \big\langle \chi_{-\theta+\theta_1} \circ \Omega^\varepsilon_\theta \big\rangle(u) \Big)^{-1} \Big\langle (\partial_u \chi_{-\theta+\theta_1} \cdot K) \circ \Omega^\varepsilon_\theta \Big\rangle(u). \tag{4.6}$$

Defining an operator which extracts the average behaviour

$$\mathcal{A}^{\theta_1}[\varphi] := \big(\partial_u \big\langle \chi_{-\theta+\theta_1} \circ \varphi_\theta \big\rangle \big)^{-1} \big\langle (\partial_u \chi_{-\theta+\theta_1} \cdot K) \circ \varphi_\theta \big\rangle, \tag{4.7}$$

the change of variable Ω^ε may be defined as the unique solution to the homological equation

$$\partial_\theta \Omega^\varepsilon_\theta - G \circ \Omega^\varepsilon_\theta = \varepsilon\big(K \circ \Omega^\varepsilon_\theta - \partial_u \Omega^\varepsilon_\theta \cdot \mathcal{A}^{\theta_1}[\Omega^\varepsilon] \big) \tag{4.8}$$

that is 1-periodic and satisfies some closure condition. Note that the above equation is considered with fixed θ_1, but modifying this phase has no impact on the definition of Ω^ε. Using the original homological equation (4.3), this may be restated as

$$\forall \theta_1 \in \mathbb{T}, \qquad K^\varepsilon = \mathcal{A}^{\theta_1}[\Omega^\varepsilon] = \mathcal{A}^0[\Omega^\varepsilon].$$

Thanks to this invariance, a group relation may be found in the case of stroboscopic averaging, summarized by the following proposition.

[1] This homological equation can also be obtained directly by considering the filtered problem of form (1.1) satisfied by $u^\varepsilon_{\theta_1}(t) = \chi_{-t/\varepsilon+\theta_1}(v^\varepsilon(t))$, which is $\partial_t u^\varepsilon_{\theta_1}(t) = f_{t/\varepsilon,\theta_1}\big(u^\varepsilon_{\theta_1}(t)\big)$.

Proposition 4.1. *If the averaging maps of the filtered problem, namely the change of variable* $(\theta, u) \mapsto \chi_{-\theta} \circ \Omega^\varepsilon_\theta(u)$ *and* $u \mapsto K^\varepsilon(u)$ *satisfy Assumption 3.1, then when considering stroboscopic averaging, for all* θ *and all* θ_0*, the following group relation is satisfied*

$$\Omega^\varepsilon_\theta \circ \Omega^\varepsilon_{\theta_0} = \Omega^\varepsilon_{\theta+\theta_0}.$$

Equivalently, there exists a vector field G^ε *such that*

$$\forall \theta,\ \forall u, \qquad \frac{\mathrm{d}}{\mathrm{d}\theta}\Omega^\varepsilon_\theta(u) = G^\varepsilon \circ \Omega^\varepsilon_\theta(u).$$

Proof. Consider the θ-map

$$\widetilde{\Omega}^\varepsilon_\theta = \Omega^\varepsilon_{\theta+\theta_0} \circ (\Omega^\varepsilon_{\theta_0})^{-1}.$$

Writing equation (4.3) with θ replaced by $\theta + \theta_0$ and $(\Omega^\varepsilon_{\theta_0})^{-1}(u)$ in lieu of u, we obtain with all maps evaluated in u,

$$\partial_\theta \widetilde{\Omega}^\varepsilon_\theta - G \circ \widetilde{\Omega}^\varepsilon_\theta = \varepsilon \left(K \circ \widetilde{\Omega}^\varepsilon_\theta - \partial_u \widetilde{\Omega}^\varepsilon_\theta \cdot \left(\partial_u (\Omega^\varepsilon_{\theta_0})^{-1}\right)^{-1} \cdot K^\varepsilon \circ (\Omega^\varepsilon_{\theta_0})^{-1} \right). \tag{4.9}$$

The new averaged vector field $\widetilde{K}^\varepsilon = \left(\partial_u (\Omega^\varepsilon_{\theta_0})^{-1}\right)^{-1} \cdot K^\varepsilon \circ (\Omega^\varepsilon_{\theta_0})^{-1}$ can be written

$$\widetilde{K}^\varepsilon = \left(\left(\partial_u \left\langle \chi_{-\theta+\theta_0} \circ \Omega^\varepsilon_\theta \right\rangle \right) \circ (\Omega^\varepsilon_{\theta_0})^{-1} \cdot \partial_u (\Omega^\varepsilon_{\theta_0})^{-1} \right)^{-1} \left\langle (\partial_u \chi_{-\theta+\theta_0} \cdot K) \circ \Omega^\varepsilon_\theta \circ (\Omega^\varepsilon_{\theta_0})^{-1} \right\rangle,$$

exploiting (4.6) with $\theta_1 = \theta_0$. The derivatives can be concatenated into $\partial_u \left\langle \chi_{-\theta+\theta_0} \circ \Omega^\varepsilon_\theta \circ (\Omega^\varepsilon_{\theta_0})^{-1} \right\rangle$. Exploiting then the phase invariance of the average, i.e. $\langle \varphi_\theta \rangle = \langle \varphi_{\theta+\theta_0} \rangle$, the identity becomes

$$\widetilde{K}^\varepsilon = \left(\partial_u \left\langle \chi_{-\theta} \circ \widetilde{\Omega}^\varepsilon_\theta \right\rangle \right)^{-1} \left\langle (\partial_u \chi_{-\theta} \cdot K) \circ \widetilde{\Omega}^\varepsilon_\theta \right\rangle = \mathcal{A}^0[\widetilde{\Omega}^\varepsilon].$$

Injecting this into (4.9), we find that $\widetilde{\Omega}^\varepsilon$ is a 1-periodic map which satisfies an equation of the form (4.8). As we only consider stroboscopic averaging, $\widetilde{\Omega}^\varepsilon$ also satisfies the same closure condition as Ω^ε, which is to say $\widetilde{\Omega}^\varepsilon_0 = \Omega^\varepsilon_0 = \mathrm{id}$. Therefore, the two maps coincide and the proof is complete.

□

Proposition 4.2. *If the averaging maps of the filtered problem, namely the change of variable* $(\theta, u) \mapsto \chi_{-\theta} \circ \Omega^\varepsilon_\theta(u)$ *and* $u \mapsto K^\varepsilon(u)$ *satisfy Assumption 3.1, then when considering stroboscopic averaging, the flows* $\theta \mapsto \Omega^\varepsilon_\theta$ *and* $t \mapsto \Psi^\varepsilon_t$ *commute with each other, i.e.*

$$\forall \theta, \quad \forall t, \qquad \Omega^\varepsilon_\theta \circ \Psi^\varepsilon_t = \Psi^\varepsilon_t \circ \Omega^\varepsilon_\theta.$$

Equivalently, the vector fields G^ε *and* K^ε *commute with each other, i.e.*

$$[G^\varepsilon, K^\varepsilon] = 0$$

where $[\cdot, \cdot]$ *is the usual Lie-bracket.*

Proof. The group law for $t \mapsto \Omega^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_t$ (recall that equation (4.1) is autonomous) gives for all s and t

$$\left(\Omega^\varepsilon_{s/\varepsilon} \circ \Psi^\varepsilon_s \right) \circ \left(\Omega^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_t \right) = \Omega^\varepsilon_{(s+t)/\varepsilon} \circ \Psi^\varepsilon_{s+t}. \tag{4.10}$$

The t-flow Ψ^ε_t satisfies a group-law by construction and owing to Proposition 4.1, this is also the case for Ω^ε_τ. Hence, we can compose equation (4.10) from the left by $\Omega^\varepsilon_{-s/\varepsilon}$ and from the right by Ψ^ε_{-t} and obtain

$$\Psi^\varepsilon_s \circ \Omega^\varepsilon_{t/\varepsilon} = \Omega^\varepsilon_{t/\varepsilon} \circ \Psi^\varepsilon_s.$$

The commutation of the vector fields then follows in a standard way. □

Note that this result can also be obtained from the proof of Proposition 4.1, since there we find

$$K^\varepsilon = \widetilde{K}^\varepsilon = \big(\partial_u(\Omega^\varepsilon_{\theta_0})^{-1}\big)^{-1} \cdot K^\varepsilon \circ (\Omega^\varepsilon_{\theta_0})^{-1},$$

i.e. K^ε is invariant when conjugated by $\Omega^\varepsilon_{\theta_0}$.

Remark 4.3. *If G is linear, then differentiating $\Phi^\varepsilon_{\theta,\theta_0}$ w.r.t. θ and taking the average generates*

$$G^\varepsilon = \left\langle \partial_u \Phi^\varepsilon_{\theta,\theta_0} \right\rangle^{-1} G \left\langle \Phi^\varepsilon_{\theta,\theta_0} \right\rangle = \big(\partial_u \Omega_0^{std} \cdot G\big) \circ \big(\Omega_0^{std}\big)^{-1}$$

if Ω^{std} is such that $\langle e^{-(\theta-\theta_0)G}\Omega^{std}\rangle = \mathrm{id}$ owing to (2.5). Furthermore the average vector field K^{std} commutes with G, thanks to the identity

$$[G, K^{std}] = [\mathbb{S}(G^\varepsilon), \mathbb{S}(K^\varepsilon)] = \mathbb{S}\big([G^\varepsilon, K^\varepsilon]\big) = 0$$

with $\mathbb{S}(F) = \big(\partial_u \Phi_0^{std} \cdot F\big) \circ \big(\Phi_0^{std}\big)^{-1}$. In other words, the change of variable $\big(\Omega_0^{std}\big)^{-1}$ transforms the perturbed vector field $G + \varepsilon K$ into $G + \varepsilon K^{std}$, where G and K^{std} commute. This corresponds to the approach of normal forms as presented in [SVM07, Chap. IX].

4.2 The case of approximations

Consider the autonomous problem (4.1) of Section 4,

$$\dot v^\varepsilon = \frac{1}{\varepsilon} G(v^\varepsilon) + K(v^\varepsilon), \qquad v^\varepsilon(0) = v_0.$$

The flow of G, denoted $(\theta, u) \mapsto \chi_\theta(u)$, is assumed 1-periodic w.r.t. θ, and we assume that for every radius ρ, the set $\mathcal{K}_\rho$ is invariant by the flow of G. Performing averaging on this problem is equivalent to performing it on the filtered problem

$$\dot u^\varepsilon(t) = \big(\partial_u \chi_{-t/\varepsilon} \cdot K\big) \circ \chi_{t/\varepsilon}(u^\varepsilon(t)), \qquad u^\varepsilon(0) = v_0.$$

The unfiltered variable is obtained as $v^\varepsilon(t) = \chi_{t/\varepsilon}(u^\varepsilon(t))$. Given an approximation $v^\varepsilon(t) = \Omega^{[n]}_{t/\varepsilon} \circ \Psi^{[n]}_t \circ \big(\Omega^{[n]}_0\big)^{-1} + \mathcal{O}(\varepsilon^{n+1})$, an approximation on u^ε of the form (3.1) is obtained by setting $\Phi^{[n]}_\theta = \chi_{-\theta} \circ \Omega^{[n]}_\theta$. Conversely, it is also possible to obtain $\Omega^{[n]}$ from working on the filtered problem, and in the case where $u \mapsto G(u)$ is non-linear, this latter approach is generally more straightforward. The defect associated to averaging on the autonomous problem is

$$\eta^{[n]}_\theta := \frac{1}{\varepsilon}\left(\partial_\theta \Omega^{[n]}_\theta - G \circ \Omega^{[n]}_\theta\right) - K \circ \Omega^{[n]}_\theta + \partial_u \Omega^{[n]}_\theta K^{[n]} \tag{4.11}$$

and the link is made with the filtered averaging with the formula

$$\eta^{[n]}_\theta = \partial_u \chi_\theta\big(\Phi^{[n]}_\theta\big) \cdot \delta^{[n]}_\theta.$$

Proposition 4.4 (Adaptation of Propositions 4.1 and 4.2)**.**
Given averaging maps $\Phi^{[n]}$ and $K^{[n]}$ which satisfy Assumption 3.2 (with $F^{[n]}$ replaced by $K^{[n]}$) and such that the associated defect $\delta^{[n]}$ satisfies (3.2), define the change of variable $(\theta, u) \mapsto \Omega^{[n]}_\theta(u) = \chi_{-\theta} \circ \Phi^{[n]}_\theta(u)$ for autonomous averaging. With this definition, there exists a vector field $u \mapsto G^{[n]}(u)$ defined on $\mathcal{K}_R$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Omega^{[n]}_\theta = G^{[n]} \circ \Omega^{[n]}_\theta + \mathcal{O}(\varepsilon^{n+1}) \quad \text{and} \quad [G^{[n]}, K^{[n]}] = \mathcal{O}(\varepsilon^{n+1}).$$

In particular, $\Omega^{[n]}$, $G^{[n]}$ and $K^{[n]}$ may be modified by terms of order ε^{n+1} to have these identities met with no error.

Proof. The first step of the proof is to show

$$K^{[n]} = \mathcal{A}^{\theta_1}[\Omega^{[n]}] + \mathcal{O}(\varepsilon^{n+1})$$

for all phases $\theta_1 \in \mathbb{T}$, with $\mathcal{A}^{\theta_1}$ the operator defined in (4.7). This result stems from the identity on $\widetilde{\Phi}_\theta^{[n]} = \chi_{-\theta-\theta_1} \circ \Omega_\theta^{[n]}$,

$$\partial_\theta \widetilde{\Phi}_\theta^{[n]} = \varepsilon\big(f_{\theta+\theta_1} \circ \widetilde{\Phi}_\theta^{[n]} - \partial_u \widetilde{\Phi}_\theta^{[n]} \cdot K^{[n]}\big) - \varepsilon \partial_u \chi_{-\theta-\theta_1}(\Omega_\theta^{[n]}) \cdot \eta_\theta^{[n]}.$$

Before taking the average, compute

$$\begin{aligned}
\partial_u \chi_{-\theta-\theta_1}(\Omega_\theta^{[n]}) \cdot \eta_\theta^{[n]} &= \partial_u \chi_{-\theta-\theta_1}\big(\chi_\theta \Phi_\theta^{[n]}\big)\partial_u \chi_\theta(\Phi_\theta^{[n]})\delta_\theta^{[n]} \\
&= \partial_u(\chi_{-\theta-\theta_1} \circ \chi_\theta \circ \Phi_\theta^{[n]})\big(\partial_u \Phi_\theta^{[n]}\big)^{-1}\delta_\theta^{[n]} \\
&= \partial_u \chi_{-\theta_1}(\Phi_\theta^{[n]})\delta_\theta^{[n]}
\end{aligned}$$

Hence this term can be written as $\partial_u \chi_{-\theta_1}\big(\mathrm{id} + \mathcal{O}(\varepsilon)\big)\delta_\theta^{[n]}$, and its average is of size $\mathcal{O}(\varepsilon^{n+1})$ thanks to the assumption on $\delta^{[n]}$. Taking the average of the previous identity, we finally obtain

$$K^{[n]} = \mathcal{A}^{\theta_1}[\Omega^{[n]}] + \mathcal{O}(\varepsilon^{n+1}).$$

We then proceed in the same manner as for the proof of Proposition 4.1. For some phase $\theta_0 \in \mathbb{T}$, consider the map $\widetilde{\Omega}_\theta^{[n]} = \Omega_{\theta+\theta_0}^{[n]} \circ \big(\Omega_{\theta_0}^{[n]}\big)^{-1}$ defined on $\mathcal{K}_R$. By definition of the defect, this new map satisfies the equation,

$$\partial_\theta \widetilde{\Omega}_\theta^{[n]} - G \circ \widetilde{\Omega}_\theta^{[n]} = \varepsilon\left(K \circ \widetilde{\Omega}_\theta^{[n]} - \partial_u \widetilde{\Omega}_\theta^{[n]} \cdot \widetilde{K}^{[n]}\right) - \varepsilon \widetilde{\eta}_\theta^{[n]}.$$

with $\widetilde{K}^{[n]} = \big(\partial_u(\Omega_{\theta_0}^{[n]})^{-1}\big)^{-1} \cdot K^{[n]} \circ (\Omega_{\theta_0}^{[n]})^{-1}$ and $\widetilde{\eta}_\theta^{[n]} = \eta_{\theta+\theta_0}^{[n]} \circ (\Omega_{\theta_0}^{[n]})^{-1}$. From (4.11), it appears in particular that $K^{[n]} = \mathcal{A}^{\theta_0}[\Omega^{[n]}] + \mathcal{O}(\varepsilon^{n+1})$. Injected into $\widetilde{K}^{[n]}$, this generates

$$\widetilde{K}^{[n]} = \Big(\partial_u \big\langle \chi_{-\theta} \circ \widetilde{\Omega}_\theta^{\varepsilon} \big\rangle\Big)^{-1} \Big\langle (\partial_u \chi_{-\theta} \cdot K) \circ \widetilde{\Omega}_\theta^{\varepsilon} \Big\rangle + \mathcal{O}(\varepsilon^{n+1}) = \mathcal{A}^0[\widetilde{\Omega}^{\varepsilon}] + \mathcal{O}(\varepsilon^{n+1}).$$

Hence $\Omega^{[n]}$ and $\widetilde{\Omega}^{[n]}$ satisfy the same equation up to a modification of the defect while still respecting (3.2). In other words, we can replace $\Omega^{[n]}$ by $\widetilde{\Omega}^{[n]}$ in the following equation

$$\partial_\theta \Omega_\theta^{[n]} - G \circ \Omega_\theta^{[n]} = \varepsilon\left(K \circ \Omega_\theta^{[n]} - \partial_u \Omega_\theta^{[n]} \cdot \mathcal{A}^0[\Omega^{[n]}]\right) + \mathcal{O}(\varepsilon^{n+1})$$

without impacting the result. Since these two maps satisfy the same closure condition $\Omega_0^{[n]} = \mathrm{id} + \mathcal{O}(\varepsilon^{n+1})$, they differ by only $\mathcal{O}(\varepsilon^{n+1})$ at any phase $\theta \in \mathbb{T}$. We can finally define

$$G^{[n]} = \partial_\theta \Omega_\theta^{[n]}\big|_{\theta=0}.$$

The second part of the theorem stems from the identity $K^{[n]} = \widetilde{K}^{[n]} + \mathcal{O}(\varepsilon^{n+1})$ which becomes

$$K^{[n]} \circ \Omega_{\theta_0}^{[n]} = \partial_u \Omega_{\theta_0}^{[n]} \cdot K^{[n]} + \mathcal{O}(\varepsilon^{n+1}).$$

□

Note that defined as such, the exact flow of $G^{[n]}$ may not be 1-periodic depending on its definition. Think for instance of the one-dimensional example $G^{[n]} = 2i\pi(1-\varepsilon)\sum_{k=0}^{n}\varepsilon^k = 2i\pi(1-\varepsilon^{n+1})$, which does not generate a 1-periodic flow while its limit $(n \to \infty)$ does.

5 Stroboscopic averaging and geometry

In this section, we start by defining some geometric properties on vector fields (namely, being divergence-free, Hamiltonian or B-Poisson, or having an invariant). We then consider a problem with forced oscillations, i.e. of the form (1.1), and show that if for all $\theta \in \mathbb{T}$, $u \mapsto f_\theta(u)$ satisfies one of the geometric properties previously presented, then the averaged vector field F^ε satisfies the same property; first in an exact setting, then when considering approximations.

This result extends to autonomous problems of the form (1.2). Indeed, if both G and K satisfy a geometric property, then this property is also satisfied by f_θ for all $\theta \in \mathbb{T}$ in the associated filtered problem, since by definition,

$$f_\theta(u) = \big(\partial_u \chi_\theta(u)\big)^{-1} \cdot K \circ \chi_\theta(u).$$

In other words, f_θ is obtained by conjugating K with χ_θ, with K in a Lie algebra and χ_θ in the associated Lie group. The algebra being stable by conjugation, f_θ satisfies the same geometric property. Since the averaged vector field of the autonomous case (4.1) coincides with the averaged vector field of the filtered case (as seen in Section 4), there is no need to distinguish the autonomous case from the case with forced oscillations.

5.1 Definitions of geometric properties

Definition 5.1. *A vector field function $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is said to be divergence free*

$$\forall u \in \mathbb{R}^d, \quad \sum_{i=1}^{d} \partial_i f_i(u) = \mathrm{tr}(\partial_u f) = 0.$$

A smooth function $(\tau, u) \in \mathbb{R} \times \mathbb{R}^d \mapsto S_\tau(u) \in \mathbb{R}^d$ is said to be volume-preserving iff

$$\forall (\tau, u) \in \mathbb{R} \times \mathbb{R}^d, \quad \det\left(\partial_u S_\tau(u)\right) = 1.$$

Remark 5.2. *By differentiation of the determinant, it is straightforward that the τ-flow of a divergence-free vector field is volume preserving. The converse is true as well.*

Definition 5.3. *A smooth functional $I : X \to \mathbb{R}$ is said to be an invariant of a vector field function $f : X \to X$ iff*

$$\forall u \in X, \quad \partial_u I(u) f(u) = 0.$$

A smooth map $(\tau, u) \in \mathbb{R} \times X \mapsto S_\tau(u)$ is said to preserve the functional I iff

$$\forall (\tau, u) \in \mathbb{R} \times X, \qquad I(u) = I \circ S_\tau(u).$$

Remark 5.4. *It follows from a straightforward τ-differentiation of $I \circ S_\tau$ that a vector field f admits an invariant if and only if its τ-flow S_τ preserves that invariant.*

Definition 5.5. *Define the matrix $J \in \mathcal{M}(\mathbb{R}^{2d})$ as the block matrix*

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

A vector field function $f : \mathbb{R}^{2d} \mapsto \mathbb{R}^{2d}$ is said to be canonically Hamiltonian if there exists a scalar smooth function $H : \mathbb{R}^{2d} \mapsto \mathbb{R}$ such that

$$\forall u \in \mathbb{R}^{2d}, \quad f(u) = J^{-1} \nabla_u H(u).$$

A smooth map $(\tau, u) \in \mathbb{R} \times \mathbb{R}^{2d} \mapsto S_\tau(u) \in \mathbb{R}^{2d}$ is said to be symplectic iff

$$\forall (\tau, u) \in \mathbb{R} \times \mathbb{R}^{2d}, \quad (\partial_u S_\tau(u))^T J (\partial_u S_\tau(u)) = J, \tag{5.1}$$

or equivalently

$$\forall (\tau, u) \in \mathbb{R} \times \mathbb{R}^{2d}, \quad (\partial_u S_\tau(u)) J^{-1} (\partial_u S_\tau(u))^T = J^{-1}. \tag{5.2}$$

Remark 5.6. *It is known that the τ-flow of a canonically Hamiltonian system is symplectic and that the converse is also true on connected sets. This is proved by differentiation and use of the integrability Lemma, which asserts that, on a connected set, a vector function derives from a gradient iff its jacobian is symmetric.*

Definition 5.7. *A matrix $B(u) \in \mathcal{M}(\mathbb{R}^d)$ is said to be a Poisson matrix if it is skew-symmetric and satisfies the Jacobi relation*

$$\forall i, j, k \in \{1, \dots, n\}, \quad \sum_{l=1}^{d} (\partial_l b_{ij}) b_{lk} + (\partial_l b_{jk}) b_{li} + (\partial_l b_{ki}) b_{lj} = 0.$$

A vector field function $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is said to be Poisson if there exists a scalar smooth function $H : \mathbb{R}^d \mapsto \mathbb{R}$ and a Poisson matrix $B(u)$ such that

$$\forall u \in \mathbb{R}^d, \quad f(u) = B(u) \nabla_u H(u).$$

A smooth function $(\tau, u) \in \mathbb{R} \times \mathbb{R}^d \mapsto S_\tau(u) \in \mathbb{R}^d$ is said to be a Poisson map iff

$$\forall u \in \mathbb{R}^d, \quad (\partial_u S_\tau(u)) B(u) (\partial_u S_\tau(u))^T = B(S_\tau(u)).$$

Remark 5.8. *The τ-flow of a Poisson system is a Poisson map, and the converse is locally true if in addition the Casimirs (spanning the null space of B) are preserved by the flow. This result is found for instance in [HLW06, Chap. VII, Thm. 4.5].*

5.2 The linear case

Theorem 5.9. *If $(\theta, u) \mapsto f_\theta(u)$ is linear w.r.t. u, then Assumption 3.1 is met, and $(\theta, u) \mapsto \Phi^\varepsilon_\theta(u)$ and $u \mapsto F^\varepsilon(u)$ are linear w.r.t. u. In that case, stroboscopic averaging is a geometric procedure. More precisely, for ε small enough, if for all $\theta \in \mathbb{T}$,*

(i) f_θ is a divergence-free vector field and X is of dimension $d < \infty$, then F^ε is also divergence-free;

(ii) the quadratic form I is an invariant of f_θ, then it is an invariant of F^ε;

(iii) f_θ is a Hamiltonian vector field, then F^ε is Hamiltonian;

(iv) f_θ is a B-Poisson vector field, then F^ε is B-Poisson.

Note that since $\Phi^\varepsilon_{t/\varepsilon} \Psi^\varepsilon_t$ is exactly the t-flow of Problem (1.1), the change of variable also has geometric properties. Every property can be proven using the following lemma:

Lemma 5.10. *At fixed $\varepsilon > 0$, consider the linear Cauchy problem*

$$\partial_t y^\varepsilon = L^\varepsilon y^\varepsilon, \qquad y^\varepsilon(0) = y_0 \in E, \tag{5.3}$$

in some Banach space E. If $|||L^\varepsilon||| < 2\pi/\varepsilon$, then $y^\varepsilon(\varepsilon) = y_0$ if and only if y^ε is constant.

Proof. If y^ε is constant, then in particular $y^\varepsilon(\varepsilon) = y_0$. Conversly, invoking the ε-periodicity of $t \mapsto y^\varepsilon(t)$, it is possible to write $y^\varepsilon(t)$ as a Fourier series, $y^\varepsilon(t) = \sum_k y_k e^{ik\frac{2\pi}{\varepsilon}t}$. The equation on y^ε can be separated into $\frac{2k\pi}{\varepsilon} y_k = L^\varepsilon y_k$ for all $k \in \mathbb{Z}$. It follows that y_k is zero for all $k \neq 0$. □

Proof. From Assumption 3.1, we set $\kappa > 0$ such that

$$|||F^\varepsilon||| \leq \kappa.$$

Writing φ_t^ε the t-flow associated with Problem (1.1), and we will often use the identity for stroboscopic times $t = \varepsilon k$ with $k \in \mathbb{Z}$,

$$\varphi_{\varepsilon k}^\varepsilon = \Psi_{\varepsilon k}^\varepsilon. \tag{5.4}$$

(i) By differentiation of the determinant,

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(\Psi_t^\varepsilon) = \mathrm{tr}(F^\varepsilon) \det(\Psi_t^\varepsilon),$$

with $\det(\Psi_\varepsilon^\varepsilon) = \det(\Psi_0^\varepsilon) = 1$ thanks to (5.4), and $|\mathrm{tr}(F^\varepsilon)| \leq d\kappa$ with d the dimension of X. From a direct application of Lemma 5.10, the determinant is constant for ε small enough, i.e. F^ε is divergence-free.

(ii) Since I is a normal form, we may find a matrix Q such that for all $u \in X$,

$$I(u) = u^T Q u. \tag{5.5}$$

Using (5.4), the quantity $I \circ \Psi_t^\varepsilon$ is preserved at stroboscopic times, i.e. $I \circ \Psi_\varepsilon^\varepsilon = I$. Furthermore, exploiting the commutativity $\Psi_t^\varepsilon F^\varepsilon = F^\varepsilon \Psi_t^\varepsilon$, a differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} (I \circ \Psi_t^\varepsilon) = \left(F^\varepsilon\right)^T I \circ \Psi_t^\varepsilon + (I \circ \Psi_t^\varepsilon) F^\varepsilon, \tag{5.6}$$

which is of the form (5.3) with $L^\varepsilon M = (F^\varepsilon)^T M + M F^\varepsilon$, of norm bounded by 2κ. For ε small enough, Lemma 5.10 can be applied, therefore I is preserved by Ψ^ε, i.e. it is an invariant of F^ε.

(iii) Considering the symplectic structure $\Psi_t^\varepsilon J^{-1} (\Psi_t^\varepsilon)^T$, the same reasoning can be conducted, therefore Ψ_t^ε is symplectic for ε small enough, i.e. F^ε is Hamiltonian.

(iv) Up to a change of variable, we assume that the Poisson matrix B is of the block form

$$B = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Lambda \end{array}\right)$$

with Λ invertible skew-symmetric. Therefore a vector field G is B-Poisson if and only if it is of the form

$$G = \left(\begin{array}{c|c} 0 & 0 \\ \hline * & \Lambda S \end{array}\right)$$

with S symmetric. From *(ii)*, we know that the Casimirs (the zero-subspace) is preserved by F^ε, therefore the top row of F^ε has to be zero. From *(iii)*, the lower-right block of F^ε generates a Λ-symplectic flow, which means this block must be of the form ΛS. Thus, F^ε is B-Poisson.

□

Remark 5.11. *It may be of interest to note that in the linear autonomous case $\partial_t u = \frac{1}{\varepsilon}Gu + Ku$, property (i) of volume-preservation does not involve the dimension. Indeed differentiating the filtered change of variable $\Phi_\theta = e^{-\theta G}e^{\theta G^\varepsilon}$ and taking the average yields*

$$G^\varepsilon = \langle\Phi\rangle^{-1} G \langle\Phi\rangle.$$

In the homological equation $\partial_\theta \Omega_\theta = \varepsilon(K\Omega_\theta - \Omega_\theta K^\varepsilon)$, we obtain

$$K^\varepsilon = \langle\Phi\rangle^{-1} K \langle\Phi\rangle.$$

The involvement of the dimension in our proof actually seems purely technical, since the averaged vector field F^ε can be expressed as a power series in ε which converges for ε small enough. Our result shows that every term of the series must be divergence-free, but the radius of convergence of the series ε_0 may be independent of the dimension of the space.

5.3 Approximations on bounded domains

Here is what the preservation of geometric properties presented in Section 5 becomes.

Theorem 5.12 (Adaptation of Theorem 5.9)**.**
Consider Assumption 3.2 met and denote φ_t^ε the t-flow associated to Problem (1.1). Up to a reduction of ε_0, the following properties are satisfied up to an error of size $\mathcal{O}(\varepsilon^{n+1})$ for all $(n+1)\varepsilon \leq \varepsilon_0$: if for all $t \in [0, T_R]$,

- *(i) $u \mapsto \varphi_t^\varepsilon(u)$ is volume-preserving on $\mathcal{K}_{2R}$, then $\Psi_t^{[n]}$ is volume-preserving on $\mathcal{K}_R$;*
- *(ii) the functional I is preserved by φ_t^ε on $\mathcal{K}_{2R}$ with $\mathcal{K}$ bounded, then it is preserved by $\Psi_t^{[n]}$ on $\mathcal{K}_R$;*
- *(iii) φ_t^ε is symplectic on $\mathcal{K}_{2R}$, then $\Psi_t^{[n]}$ is symplectic on $\mathcal{K}_R$;*
- *(iv) φ_t^ε is B-symplectic and preserves Casimirs on $\mathcal{K}_{2R}$, then $\Psi_t^{[n]}$ is B-symplectic and preserves Casimirs on $\mathcal{K}_R$.*

Note that since $\Phi_\theta^{[n]} = \varphi_{\varepsilon\theta}^\varepsilon \circ \Psi_{-\varepsilon\theta}^{[n]} + \mathcal{O}(\varepsilon^{n+1})$, these properties are also true for $\Phi_\theta^{[n]}$, up to terms of size $\mathcal{O}(\varepsilon^{n+1})$. It is therefore possible to modify $\Phi^{[n]}$ and $F^{[n]}$ and have these properties met exactly.

Proof. As can be seen in the proof of Theorem 5.9, every property can be proven in the same way. Therefore we will only describe how to prove *(iii)*, as it is probably the most interesting property for the majority of readers. We refer to the other proof for the adaptation to other properties.

Set $(t, u) \mapsto \Delta_t(u)$ the deviation from symplecticity,

$$\Delta_t = \big(\partial_u \Psi_t^{[n]}\big) J^{-1} \big(\partial_u \Psi_t^{[n]}\big)^T - J^{-1},$$

defined and bounded for $u \in \mathcal{K}_{R_n}$ Thanks the periodicity of $\Phi^{[n]}$, $t \mapsto \Delta_t$ is almost zero at stroboscopic times, meaning that for all $k \in \mathbb{N}$ such that $\varepsilon k \leq T_R$, since $\Psi_{\varepsilon k}^{[n]} = \varphi_{\varepsilon k}^\varepsilon + \mathcal{O}(\varepsilon^{n+1})$,

$$\Delta_{\varepsilon k} = \big(\partial_u \varphi_{\varepsilon k}^\varepsilon\big) J^{-1} \big(\partial_u \varphi_{\varepsilon k}^\varepsilon\big)^T - J^{-1} + \mathcal{O}(\varepsilon^{n+1}) = \mathcal{O}(\varepsilon^{n+1}).$$

For now let us conduct our reasoning on $(t, u) \in [0, \varepsilon] \times \mathcal{K}_R$. Setting $L_t M = \partial_u F^{[n]}(\Psi_t^{[n]}) M + M\big(\partial_u F^{[n]}(\Psi_t^{[n]})\big)^T$ and $S_t = L_t J^{-1}$, it satisfies

$$\partial_t \Delta_t = L_t\, \Delta_t + S_t, \quad \text{i.e.} \quad \Delta_t = \Delta_0 + \int_0^t L_\tau\, \Delta_\tau \mathrm{d}\tau + \int_0^t S_\tau\, \mathrm{d}\tau. \tag{5.7}$$

We want to prove $\sup_{0\le t\le\varepsilon}\|\Delta_t\|_R = \mathcal{O}(\varepsilon^{n+1})$. To that effect, introduce the norm $\|\cdot\|_{\varepsilon,\rho}$ and the radii R_k,

$$\|g\|_{\varepsilon,\rho} = \sup_{0\le t\le\varepsilon}\|g_t\|_\rho \qquad \text{and} \qquad R_k = R + kr_n \quad \text{with} \quad r_n = \frac{R}{n+1},$$

and set $\alpha > 0$ such that $\|\Delta_0\|_{2R}, \|\Delta_\varepsilon\|_{2R} \le \alpha\varepsilon^{n+1}$. Gronwall's lemma in the integral form of Δ_t yields

$$\|\Delta\|_{\varepsilon,R} \le \big(\alpha\varepsilon^{n+1} + \varepsilon\|S\|_{\varepsilon,R}\big)e^{\varepsilon\|L\|_{\varepsilon,R}}, \tag{5.8}$$

therefore we want to show $\|S\|_{\varepsilon,R} = \mathcal{O}(\varepsilon^n)$ so that $\Delta_t = \mathcal{O}(\varepsilon^{n+1})$. Because S is transported by $F^{[n]}$, i.e. $S_t = S_0 \circ \Psi_t^{[n]}$, it is possible to bound S_t on some space $\mathcal{K}_\rho$ by the norm of S_0 on a larger space. In particular, assuming $\varepsilon_0 \le R/(2M)$, i.e. $\varepsilon \le r_n/(2M)$,

$$\|S\|_{\varepsilon,R_k} \le \|S_0\|_{R_k + r_n/2} \le \|S_0\|_{R_{k+1}} \tag{5.9}$$

since $\|\Psi_t^{[n]} - \mathrm{id}\,\|_{R_n} \le tM$. Additionally from (5.7) evaluated at $t = \varepsilon$, we gather

$$\left|\int_0^\varepsilon S_t \mathrm{d}t\right| \le 2\alpha\varepsilon^{n+1} + \int_0^\varepsilon \|L_t\|_R \mathrm{d}t\, \|\Delta\|_{\varepsilon,R}. \tag{5.10}$$

An integration by parts transforms the left integral, $\int_0^\varepsilon S_t\mathrm{d}t = \varepsilon S_0 + \int_0^\varepsilon(\varepsilon - t)\partial_t S_t \mathrm{d}t$, and because S_t is transported by $F^{[n]}$, its derivative w.r.t. t may be bounded as

$$\|\partial_t S_t\|_R = \left\|(\partial_u S_0 \cdot F^{[n]}) \circ \Psi_t^\varepsilon\right\|_R \le \left\|\partial_u S_0 \cdot F^{[n]}\right\|_{R+r_n/2}$$

We may then use a so-called *Cauchy estimate* : since the function $u \mapsto S_0(u)$ is analytic around any $u \in \mathcal{K}_{R+r_n/2}$ with a radius of convergence $\rho > R$, then for all $0 < \delta \le R$,

$$\partial_u S_0(u)\cdot v = \frac{1}{2i\pi}\int_{|\xi|=\delta/|v|} \frac{S_0(u+\xi v)}{\xi^2}\mathrm{d}\xi,$$

where $S_0(u + \xi v)$ is defined by the power series around u. From this we deduce the so-called Cauchy estimate,

$$\big|\partial_u S_0(u)\cdot v\big| \le \frac{|v|}{\delta}\sup_{|\xi|=\delta/|v|}|S_0(u+\xi v)|. \tag{5.11}$$

In particular with $\delta = r_n/2$, we obtain

$$\|\partial_t S_t\|_R \le \left\|\partial_u S_0 \cdot F^{[n]}\right\|_{R+r_n/2} \le \frac{M}{r_n/2}\|S_0\|_{R_1} \le \frac{1}{\varepsilon_n}\|S_0\|_{R_1}.$$

Using the integration by parts along with this estimate in (5.10), we obtain

$$\varepsilon\|S_0\|_R \le \frac{\varepsilon^2}{2\varepsilon_n}\|S_0\|_{R_1} + 2\alpha\varepsilon^{n+1} + \int_0^\varepsilon \|L_t\|_R \mathrm{d}t\, \|\Delta\|_{\varepsilon,R}.$$

Since L_t is also transported by $F^{[n]}$, the integral can also be bounded, $\int_0^\varepsilon \|L_t\|_R\mathrm{d}t \le \varepsilon\|L\|_{\varepsilon,R} \le \varepsilon\|L_0\|_{R+r_n/2}$, and by definition along with a Cauchy estimate,

$$\|L_0\|_{R+r_n/2} \le 2\|\partial_u F^\varepsilon\|_{R+r_n/2} \le \frac{1}{r_n/2}M \le \frac{1}{\varepsilon_n}.$$

This bound, as well as (5.8) can be injected into (5.3) to yield

$$\|S_0\|_R \leq \frac{\varepsilon}{\varepsilon_n}\left(\frac{1}{2} + e^{\varepsilon/\varepsilon_n}\right)\|S_0\|_{R_1} + (2 + e^{\varepsilon/\varepsilon_n}\varepsilon/\varepsilon_n)\alpha\varepsilon^n \leq \frac{4\varepsilon}{\varepsilon_n}\|S_0\|_{R_1} + 5\alpha\varepsilon^n.$$

The same reasoning can be conducted on any $\mathcal{K}_{R_k}$ to obtain

$$\|S_0\|_{R_k} \leq \frac{4\varepsilon}{\varepsilon_n}\|S_0\|_{R_{k+1}} + 5\alpha\varepsilon^n$$

therefore, by successive injections,

$$\|S_0\|_R \leq \left(\frac{4\varepsilon}{\varepsilon_n}\right)^{n+1}\|S_0\|_{2R} + 5\alpha\varepsilon^n\sum_{k=0}^{n}\left(\frac{4\varepsilon}{\varepsilon_n}\right)^k$$

therefore, because $\|S_0\|_{2R}$ is bounded by definition, if ε is small enough,

$$\|S_0\|_{R_1} = \mathcal{O}(\varepsilon^n), \qquad \text{and in turn} \qquad \|\Delta\|_{\varepsilon,R} = \mathcal{O}(\varepsilon^{n+1}).$$

This result is also true on all intervals of the form $[k\varepsilon, (k+1)\varepsilon]$, therefore $(t, u) \mapsto \Psi_t^\varepsilon(u)$ is symplectic up to terms of order $\mathcal{O}(\varepsilon^{n+1})$.

□

References

[Bam03] Dario Bambusi. "Birkhoff normal form for some nonlinear PDEs". In: *Communications in mathematical physics* 234.2 (2003). Publisher: Springer, pp. 253–285.

[Bam06] Dario Bambusi. "Birkhoff normal form for some quasilinear Hamiltonian PDEs". In: *XIVth International Congress on Mathematical Physics*. World Scientific, 2006, pp. 273–280.

[Bam08] Dario Bambusi. "A Birkhoff normal form theorem for some semilinear PDEs". In: *Hamiltonian dynamical systems and applications*. Springer, 2008, pp. 213–247.

[BB05] Dario Bambusi and Massimiliano Berti. "A Birkhoff–Lewis-Type Theorem for Some Hamiltonian PDEs". In: *SIAM Journal on Mathematical Analysis* 37.1 (2005). Publisher: SIAM, pp. 83–102.

[BCZ14] Weizhu Bao, Yongyong Cai, and Xiaofei Zhao. "A Uniformly Accurate Multiscale Time Integrator Pseudospectral Method for the Klein–Gordon Equation in the Nonrelativistic Limit Regime". In: *SIAM Journal on Numerical Analysis* 52.5 (Jan. 2014), pp. 2488–2511.

[BZ19] Weizhu Bao and Xiaofei Zhao. "Comparison of numerical methods for the nonlinear Klein-Gordon equation in the nonrelativistic limit regime". In: *Journal of Computational Physics* 398 (Dec. 2019), p. 108886.

[Bou96] J. Bourgain. "Construction of approximative and almost periodic solutions of perturbed linear schrödinger and wave equations". In: *Geometric & Functional Analysis GAFA* 6.2 (Mar. 1996), pp. 201–230.

[Bri26] Léon Brillouin. "Remarques sur la mécanique ondulatoire". In: *J. phys. radium* 7.12 (1926), pp. 353–368.

[CCM19] Fernando Casas, Philippe Chartier, and Ander Murua. "Continuous changes of variables and the Magnus expansion". In: *Journal of Physics Communications* 3.9 (Sept. 2019), p. 095014.

[CCMM15] F. Castella, Ph. Chartier, F. Méhats, and A. Murua. "Stroboscopic Averaging for the Nonlinear Schrödinger Equation". In: *Foundations of Computational Mathematics* 15.2 (Apr. 2015), pp. 519–559.

[CMS10] P. Chartier, A. Murua, and J. M. Sanz-Serna. “Higher-Order Averaging, Formal Series and Numerical Integration I: B-series”. In: *Foundations of Computational Mathematics* 10.6 (Dec. 2010), pp. 695–727.

[CCLM15] Philippe Chartier, Nicolas Crouseilles, Mohammed Lemou, and Florian Méhats. “Uniformly accurate numerical schemes for highly oscillatory Klein–Gordon and nonlinear Schrödinger equations”. In: *Numerische Mathematik* 129.2 (2015). Publisher: Springer, pp. 211–250.

[CCLM20] Philippe Chartier, Nicolas Crouseilles, Mohammed Lemou, and Florian Méhats. “Averaging of highly-oscillatory transport equations”. In: *Kinetic & Related Models* 13.6 (2020), p. 1107.

[CCLMZ20] Philippe Chartier, Nicolas Crouseilles, Mohammed Lemou, Florian Méhats, and Xiaofei Zhao. “Uniformly Accurate Methods for Three Dimensional Vlasov Equations under Strong Magnetic Field with Varying Direction”. In: *SIAM Journal on Scientific Computing* 42.2 (Jan. 2020), B520–B547.

[CLMV20] Philippe Chartier, Mohammed Lemou, Florian Méhats, and Gilles Vilmart. “A New Class of Uniformly Accurate Numerical Schemes for Highly Oscillatory Evolution Equations”. In: *Foundations of Computational Mathematics* 20.1 (Feb. 2020), pp. 1–33.

[CLMZ20] Philippe Chartier, Mohammed Lemou, Florian Méhats, and Xiaofei Zhao. “Derivative-free high-order uniformly accurate schemes for highly-oscillatory systems”. In: *submitted preprint* (2020).

[CMTZ17] Philippe Chartier, Florian Méhats, Mechthild Thalhammer, and Yong Zhang. “Convergence of multi-revolution composition time-splitting methods for highly oscillatory differential equations of Schrödinger type”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 51.5 (Sept. 2017), pp. 1859–1882.

[CMS12a] Philippe Chartier, Ander Murua, and Jesus Maria Sanz-Serna. “A formal series approach to averaging: exponentially small error estimates”. In: *Discrete and Continuous Dynamical Systems-Series A* 32.9 (2012).

[CMS12b] Philippe Chartier, Ander Murua, and Jesus Maria Sanz-Serna. “Higher-order averaging, formal series and numerical integration II: the quasi-periodic case”. In: *Foundations of Computational Mathematics* 12.4 (2012). Publisher: Springer, pp. 471–508.

[CMS15] Philippe Chartier, Ander Murua, and Jesus Maria Sanz-Serna. “Higher-order averaging, formal series and numerical integration III: error bounds”. In: *Foundations of Computational Mathematics* 15.2 (2015). Publisher: Springer, pp. 591–612.

[CKSTT10] James Colliander, Markus Keel, Gigiola Staffilani, Hideo Takaoka, and Terence Tao. “Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation”. In: *Inventiones mathematicae* 181.1 (2010). Publisher: Springer, pp. 39–113.

[CKO12] James Colliander, Soonsik Kwon, and Tadahiro Oh. “A remark on normal forms and the “upside-down” I-method for periodic NLS: growth of higher Sobolev norms”. In: *Journal d’Analyse Mathématique* 118.1 (2012). Publisher: Springer, pp. 55–82.

[FS00] E. Frénod and E. Sonnendrücker. “Long time behavior of the two-dimensional vlasov equation with a strong external magnetic field”. In: *Mathematical Models and Methods in Applied Sciences* 10.4 (June 2000), pp. 539–553.

[FSS09] Emmanuel Frénod, Francesco Salvarani, and Eric Sonnendrücker. “Long time simulation of a beam in a periodic focusing channel via a two-scale pic-method”. In: *Mathematical Models and Methods in Applied Sciences* 19.2 (Feb. 2009), pp. 175–197.

[GSS98] Bosco Garcia-Archilla, Jesús Maria Sanz-Serna, and Robert D Skeel. “Long-time-step methods for oscillatory differential equations”. In: *SIAM Journal on Scientific Computing* 20.3 (1998). Publisher: SIAM, pp. 930–963.

[GT12] Benoit Grébert and Laurent Thomann. “Resonant dynamics for the quintic nonlinear Schrödinger equation”. In: *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*. Vol. 29. Issue: 3. Elsevier, 2012, pp. 455–477.

[GV11] Benoit Grébert and Carlos Villegas-Blas. “On the energy exchange between resonant modes in nonlinear Schrödinger equations”. In: *Annales de l’Institut Henri Poincaré C, Analyse non linéaire* 28.1 (Jan. 2011), pp. 127–134.

[HLW06] Ernst Hairer, Christian Lubich, and Gerhard Wanner. *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*. 2nd ed. Springer Series in Computational Mathematics. Berlin Heidelberg: Springer-Verlag, 2006.

[Kra26] Hendrik Anthony Kramers. “Wellenmechanik und halbzahlige Quantisierung”. In: *Zeitschrift für Physik* 39.10 (1926). Publisher: Springer, pp. 828–840.

[LM88] P. Lochak and C. Meunier. *Multiphase Averaging for Classical Systems: With Applications to Adiabatic Theorems*. Applied Mathematical Sciences. New York: Springer-Verlag, 1988.

[Nei84] A. I. Neishtadt. “The separation of motions in systems with rapidly rotating phase”. In: *Journal of Applied Mathematics and Mechanics* 48.2 (Jan. 1984), pp. 133–139.

[Per69] Lawrence M. Perko. “Higher Order Averaging and Related Methods for Perturbed Periodic and Quasi-Periodic Systems”. In: *SIAM Journal on Applied Mathematics* 17.4 (July 1969), pp. 698–724.

[SVM07] Jan A. Sanders, Ferdinand Verhulst, and James Murdock. *Averaging Methods in Nonlinear Dynamical Systems*. 2nd ed. Applied Mathematical Sciences. New York: Springer-Verlag, 2007.

[Wen26] Gregor Wentzel. “Eine verallgemeinerung der quantenbedingungen für die zwecke der wellenmechanik”. In: *Zeitschrift für Physik* 38.6 (1926). Publisher: Springer, pp. 518–529.